Cascade High Gain Predictors for a Class of Nonlinear Systems

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Abstract—**This work presents a set of cascade high gain predictors to reconstruct the vector state of triangular nonlinear systems with delayed output. By using a Lyapunov-Krasvoskii approach, simple sufficient conditions ensuring the exponential convergence of the observation error towards zero are given. All predictors used in the cascade have the same structure. This feature will greatly improve the easiness of their implementation. This result is illustrated by some simulations.**

Index Terms—**Cascade systems, high gain observer, time-delay systems.**

I. Introduction

In this technical note, the design of nonlinear observers for nonlinear systems with delayed output measurements is investigated. This problem appears in many control systems areas, such as networked control systems, where the data are transmitted through a communication

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channel which introduces a delay between the sensors and observers. Note that in the linear case and when the delay is constant, this problem has been solved by the well-known *Smith predictor* [1]. In the nonlinear case, only a few works can be found in the literature. We can mention [2] where a solution based on chained observers has been proposed for a class of nonlinear systems. The authors used *Gronwall* lemma, to derive sufficient conditions on the delay and on the number of observers in the chain which guarantee exponential convergence of the observation error. The structure of the observers used in the chain is designed recursively and grows step by step. After this, some restrictive conditions of the above approach have been relaxed in [3]. More recently, in [4] another predictor for linear and nonlinear systems has been presented. This predictor is a set of cascade observers. Sufficient conditions based on *linear matrix inequalities* are derived to guarantee the asymptotic convergence of this predictor. In [5], the authors developed a cascade predictor for any constant delay based on the high gain observer framework developed in [6], [7], and extended to finite time observer in [8]–[10]. The authors of [5] proposed, for specific high gain observer (with specific vector gains), simple relations between the delay and the number of observers in the cascade. The result of [5] has been used in [11] to derive an output feedback controller for a class of nonlinear systems with delayed input. In [12], the authors showed that the results contained in [5] can be extended to time varying-delay and sampled-data cases only for one observer and if the delay is sufficiently small. In the present technical note, we will generalize the results contained in [5] and [12] by using more general observer gains which guarantee exponential convergence in the free-delay case. More precisely, we derive an explicit number of observers depending on the delay and on the gains of the observers to ensure exponential convergence of the estimated state at time t towards the true state at time t in the presence of any constant measurement delay. Compared to [2] and [3], this work can be viewed as an alternative method which uses only one structure of observers to reconstruct the systems states. This technical note is organized as follows : In the next section, we present the class of considered systems and the different assumptions. In the third one, we present the proposed observers and derive the conditions of their convergence. These results are illustrated in the last section, throughout an academic example.

II. Preliminaries and Notations

First some mathematical notations which will be used throughout the technical note are introduced.

- The euclidian norm on $\mathbb{R}^n$ will be denoted by $\|.\|$.
- The matrix X^T represents the transposed matrix of X.
- $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimum and maximum eigenvalues of the square matrix P.
- I_n is the (n,n) identity matrix.

In this technical note, we consider the following class of nonlinear systems:

$$\begin{aligned} \dot{x} &= Ax + \phi(x,u) \\ y &= Cx(t-\tau) \end{aligned} \tag{1}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \dots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \tag{2}$$

$$C = (\,1 \quad 0 \quad \dots \quad 0\,) \tag{3}$$

and

$$\phi(x,u) = \begin{pmatrix} \phi_1(x,u) \\ \vdots \\ \phi_n(x,u) \end{pmatrix}. \tag{4}$$

The term τ represents the measurement time delay which is supposed known, $x(t) = (x^1, \dots, x^n) \in \mathbb{R}^n$ is the vector state which is supposed unavailable. The output $y(t) \in \mathbb{R}$ is a linear function of the state x at time $t-\tau$. The input u belongs to U which is a compact set in $\mathbb{R}$. The functions $\phi_i, \quad i = 1, \dots, n$ are supposed smooth. Throughout the technical note, we assume that the following hypotheses are satisfied:

$\mathcal{H}1$. The functions $\phi_i(x,u)$ are triangular in x, i.e:

$$\frac{\partial \phi_i(x,u)}{\partial x^{k+1}} = 0 \quad k = i, \dots, n-1. \tag{5}$$

$\mathcal{H}2$. The functions $\phi_i(x,u)$ are globally Lipschitz, uniformly in u with respect to x, i.e.

$$\begin{aligned} &\exists \beta > 0 \ \text{ such that } \forall (x,x') \in \mathbf{R^n} \times \mathbf{R^n}, \quad \forall u \in U, \\ &\qquad \|\phi_i(x,u) - \phi_i(x',u)\| \le \beta \|x - x'\| \ \ i = 1, \dots, n. \end{aligned} \tag{6}$$

Note that the vector $\phi(x,u)$ may contain also linear parts which are not represented by the term Ax.

This class represents the class of uniformly observable systems. It has been shown [6], [7] that these models concern a wide variety of systems, such as bioreactors\cdots

III. Main Results

Let us consider an arbitrary constant time delay τ affecting the output measurement of system (1). The proposed predictor for system (1) is a set of m cascade high gain observers which use delayed output. Each of them estimates a delayed state vector with a smaller delay τ/m.

In order to present the proposed predictor, we use the following convenient notations adopted from [2]:

$$x_j(t) = x\left(t - \tau + j\frac{\tau}{m}\right) \quad \text{and} \quad u_j(t) = u\left(t - \tau + j\frac{\tau}{m}\right)$$

where $j = 1, \dots, m$.

Then the proposed predictor can be written in the following form, for $j = 1, \dots, m$:

$$\begin{cases} \dot{\hat{x}}_1 = A\hat{x}_1 + \phi(\hat{x}_1, u_1) - \theta\Delta^{-1}K\left(C\hat{x}_1\left(t - \frac{\tau}{m}\right) - y(t)\right) \\ \hat{y}_1(t) = C\hat{x}_1(t) \\ \vdots = \vdots \\ \dot{\hat{x}}_j = A\hat{x}_j + \phi(\hat{x}_j, u_j) - \theta\Delta^{-1}K\left(C\hat{x}_j\left(t - \frac{\tau}{m}\right) - \hat{y}_{j-1}(t)\right) \\ \hat{y}_j(t) = C\hat{x}_j(t) \end{cases} \tag{7}$$

where θ is a positive constant satisfying $\theta \ge 1$, the vector $K = (k_1, \dots, k_n)^T$ is chosen so that the matrix $A - KC$ is Hurwitz and Δ is a diagonal matrix which has the following form :

$$\Delta = Diag\left(1, \dots, \frac{1}{\theta^{i-1}}, \dots, \frac{1}{\theta^{n-1}}\right). \tag{8}$$

We will show that the vector $\hat{x}_j(t)$ estimates the delayed state $x_j(t)$, $j = 1, \dots, m-1$ and $\hat{x}_m(t)$ estimates $x(t)$.

Remark 1: As we can see all observers used in (7) have the same structure. This property will greatly improve the easiness of implementation.

Before proving the exponential convergence of the proposed cascade observers, we consider the case when the delay is sufficiently small and can be time-varying. Then only one high gain observer is required to estimate the state of system (1).

Lemma 1: Consider the following observer:

$$\dot{\hat{x}} = A\hat{x} + \phi(\hat{x}, u) - \theta\Delta^{-1}KC(\hat{x}(t-\tau(t)) - x(t-\tau(t))) \quad (9)$$

and suppose that the delay $\tau(t)$ is time-varying, piecewise-continuous and bounded. Then for sufficiently large θ, there exists a positive constant τ_1 such that $\forall\tau(t) \in [0, \tau_1]$, observer (9) converges exponentially towards system (1).

Proof: First let us denote the observation error as $\tilde{x} = \hat{x} - x$. Then we will have

$$\dot{\tilde{x}} = A\tilde{x} + \phi(\hat{x}, u) - \phi(x, u) - \theta\Delta^{-1}KC\tilde{x}(t-\tau(t)). \quad (10)$$

If we apply the relation

$$\tilde{x}(t) = \tilde{x}(t-\tau(t)) + \int_{t-\tau(t)}^{t} \dot{\tilde{x}}(s)ds \quad (11)$$

then we have

$$\begin{aligned}\dot{\tilde{x}} &= (A - \theta\Delta^{-1}KC)\tilde{x} \\ &\quad + \phi(\hat{x}, u) - \phi(x, u) + \theta\Delta^{-1}KC\int_{t-\tau(t)}^{t} \dot{\tilde{x}}(s)ds. \quad (12)\end{aligned}$$

Let us consider the change of coordinates $\bar{x} = \Delta\tilde{x}$, and using the fact that $\Delta A\Delta^{-1} = \theta A$ and $C\Delta = C\Delta^{-1} = C$, then system (12) can be rewritten in the following manner:

$$\begin{aligned}\dot{\bar{x}} &= \theta(A - KC)\bar{x} + \Delta(\phi(\hat{x}, u) - \phi(x, u)) \\ &\quad + \theta KC\int_{t-\tau(t)}^{t} \dot{\bar{x}}(s)ds. \quad (13)\end{aligned}$$

In order to derive an upper bound τ_1 for the delay $\tau(t)$, to ensure the exponential convergence to zero of the error $\bar{x}$, we use the following *Lyapunov-Krasovskii functional* [13]:

$$W(\bar{x}) = \bar{x}^T P\bar{x} + \int_{t-\tau_1}^{t}\int_{s}^{t} \|\dot{\bar{x}}(\xi)\|^2 d\xi ds \quad (14)$$

where the matrix P is a symmetric positive definite matrix, solution of the following algebraic Lyapunov equation:

$$P(A - KC) + (A - KC)^T P \leq -\mu I_n \quad \mu > 0. \quad (15)$$

The functional (14) can be written after some manipulations as follows:

$$W(\bar{x}) = \bar{x}^T P\bar{x} + \int_{t-\tau_1}^{t} (s - t + \tau_1)\|\dot{\bar{x}}(s)\|^2 ds. \quad (16)$$

If we compute its time derivative, we obtain

$$\begin{aligned}\dot{W} &\leq \theta\bar{x}^T(P(A - KC) + (A - KC)^T P)\bar{x} \\ &\quad + 2\bar{x}^T P\Delta(\phi(\hat{x}, u) - \phi(x, u)) \\ &\quad + 2\theta\bar{x}^T PKC\int_{t-\tau(t)}^{t} \dot{\bar{x}}(s)ds \\ &\quad + \tau_1\|\dot{\bar{x}}(t)\|^2 - \int_{t-\tau_1}^{t} \|\dot{\bar{x}}(s)\|^2 ds. \quad (17)\end{aligned}$$

Using (15), we have

$$\begin{aligned}\dot{W} &\leq -\mu\theta\|\bar{x}\|^2 + 2\bar{x}^T P\Delta(\phi(\hat{x}, u) - \phi(x, u)) \\ &\quad + 2\theta\bar{x}^T PKC\int_{t-\tau(t)}^{t} \dot{\bar{x}}(s)ds \\ &\quad + \tau_1\|\dot{\bar{x}}(t)\|^2 - \int_{t-\tau_1}^{t} \|\dot{\bar{x}}(s)\|^2 ds. \quad (18)\end{aligned}$$

with

$$\Delta\left(\phi(\hat{x}, u) - \phi(x, u)\right) = \begin{pmatrix} \phi_1(\hat{x}, u) - \phi_1(x, u) \\ \vdots \\ \frac{1}{\theta^{i-1}}(\phi_i(\hat{x}, u) - \phi_i(x, u)) \\ \frac{1}{\theta^{n-1}}(\phi_n(\hat{x}, u) - \phi_n(x, u)) \end{pmatrix}. \quad (19)$$

Using the Lipschitz property, then we can write:

$$\left\|\frac{1}{\theta^{i-1}}(\phi_i(\hat{x}, u) - \phi_i(x, u))\right\| \leq \frac{\beta}{\theta^{i-1}}\sqrt{\sum_{k=1}^{i}(\hat{x}^k - x^k)^2}. \quad (20)$$

Using the fact that $\theta \geq 1$, then we deduce that

$$\left\|\frac{1}{\theta^{i-1}}(\phi_i(\hat{x}, u) - \phi_i(x, u))\right\| \leq \beta\sqrt{\sum_{k=1}^{i}(\bar{x}^k)^2} \quad (21)$$

where $(\hat{x}^k - x^k) = \theta^{k-1}\bar{x}^k$. From this, and since $\sqrt{\sum_{k=1}^{i}(\bar{x}^k)^2} \leq \|\bar{x}\|$, then we can write

$$\left\|\frac{1}{\theta^{i-1}}(\phi_i(\hat{x}, u) - \phi_i(x, u))\right\| \leq \beta\|\bar{x}\| \quad i = 1, \ldots, n \quad (22)$$

and

$$\|\Delta\left(\phi(\hat{x}, u) - \phi(x, u)\right)\|^2 \leq n\beta^2\|\bar{x}\|^2. \quad (23)$$

This leads to

$$\|2\bar{x}^T P\Delta(\phi(\hat{x}, u) - \phi(x, u))\| \leq 2\lambda_{\max}(P)\sqrt{n}\beta\|\bar{x}\|^2. \quad (24)$$

From this, we will have

$$\begin{aligned}\dot{W} &\leq -\mu\theta\|\bar{x}\|^2 + 2\lambda_{\max}(P)\sqrt{n}\beta\|\bar{x}\|^2 \\ &\quad + 2\theta\bar{x}^T PKC\int_{t-\tau}^{t} \dot{\bar{x}}(s)ds \\ &\quad + \tau_1\|\dot{\bar{x}}(t)\|^2 - \int_{t-\tau_1}^{t} \|\dot{\bar{x}}(s)\|^2 ds. \quad (25)\end{aligned}$$

Now, let us remark that from (13) and by using Hölder's inequality, it comes

$$\begin{aligned}\|\dot{\bar{x}}(t)\|^2 &\leq 2\theta^2[\|A - KC\| + 2\lambda_{\max}(P)\sqrt{n}\beta]^2\|\bar{x}\|^2 \\ &\quad + 2\|K\|^2\eta(t)^2 \quad (26)\end{aligned}$$

where $\eta(t) = \theta\int_{t-\tau(t)}^{t} \dot{\bar{x}}(s)ds$.

Using Young inequality, then we obtain

$$\begin{aligned}&\left\|2\theta\bar{x}^T PKC\int_{t-\tau(t)}^{t} \dot{\bar{x}}(s)ds\right\| \\ &\qquad \leq \frac{\theta\mu}{2}\|\bar{x}\|^2 + \frac{2}{\mu\theta}\lambda_{\max}^2(P)\|K\|^2\|\eta(t)\|^2. \quad (27)\end{aligned}$$

Using this and (25), then we will have

$$\begin{aligned}\dot{W} \leq & -\frac{\mu\theta}{2}\|\bar{x}\|^2 + 2\lambda_{\max}(P)\sqrt{n}\beta\|\bar{x}\|^2 \\ & + \frac{2}{\mu\theta}\lambda_{\max}^2(P)\|K\|^2\|\eta(t)\|^2 \\ & + 2\tau_1\theta^2[\|A-KC\| + 2\lambda_{\max}(P)\sqrt{n}\beta]^2\|\bar{x}\|^2 \\ & + 2\tau_1\|K\|^2\|\eta(t)\|^2 - \int_{t-\tau_1}^{t}\|\dot{\bar{x}}(s)\|^2 ds. \end{aligned} \tag{28}$$

To prove the above Lemma 1, it is sufficient to find conditions which guarantee the following inequality

$$\dot{W} + \epsilon W < 0 \quad \text{with} \quad \forall \epsilon > 0. \tag{29}$$

Indeed, from the above inequality and the definition of (14), we can write

$$\bar{x}^T P \bar{x} \leq W(\bar{x}) \leq W(\bar{x}(t_0))e^{-\epsilon(t-t_0)}. \tag{30}$$

Then we deduce that

$$\|\bar{x}(t)\| \leq \frac{\sqrt{W(\bar{x}(t_0))}}{\sqrt{\lambda_{\min}(P)}}e^{-(\epsilon/2)(t-t_0)}. \tag{31}$$

Let us be interested again in (29). From (28), we can write

$$\begin{aligned}\dot{W} + \epsilon W \leq & -\frac{\mu\theta}{2}\|\bar{x}\|^2 + 2\lambda_{\max}(P)\sqrt{n}\beta\|\bar{x}\|^2 \\ & + \frac{2}{\mu\theta}\lambda_{\max}^2(P)\|K\|^2\|\eta(t)\|^2 + \epsilon\lambda_{\max}(P)\|\bar{x}\|^2 \\ & + 2\tau_1\theta^2[\|A-KC\| + 2\lambda_{\max}(P)\sqrt{n}\beta]^2\|\bar{x}\|^2 \\ & + 2\tau_1\|K\|^2\|\eta(t)\|^2 \\ & + \epsilon\tau_1\int_{t-\tau_1}^{t}\|\dot{\bar{x}}(s)\|^2 ds - \int_{t-\tau_1}^{t}\|\dot{\bar{x}}(s)\|^2 ds. \end{aligned} \tag{32}$$

If we use the *Jensen's* inequality, we derive

$$\int_{t-\tau_1}^{t}\|\dot{\bar{x}}(s)\|^2 ds \geq \frac{1}{\theta^2\tau_1}\|\eta(t)\|^2. \tag{33}$$

From this, we deduce that

$$\begin{aligned}\dot{W} + \epsilon W \leq & -\left\{\frac{\mu\theta}{2} - 2\lambda_{\max}(P)\sqrt{n}\beta \right. \\ & - 2\tau_1\theta^2[\|A-KC\| + 2\lambda_{\max}(P)\sqrt{n}\beta]^2 \\ & \left. - \epsilon\lambda_{\max}(P)\right\}\|\bar{x}\|^2 \\ & - \left\{1 - 2\lambda_{\max}^2(P)\|K\|^2\frac{\tau_1\theta}{\mu} \right. \\ & \left. - 2\tau_1^2\theta^2\|K\|^2 - \epsilon\tau_1\right\}\int_{t-\tau_1}^{t}\|\dot{\bar{x}}(s)\|^2 ds. \end{aligned} \tag{34}$$

Then, the exponential convergence to zero of the observation error $\bar{x}$ is guaranteed if the following inequalities hold:

$$\begin{cases} \frac{\mu\theta}{2} - 2\lambda_{\max}(P)\sqrt{n}\beta - 2\tau_1\theta^2[\|A-KC\| + 2\lambda_{\max}(P)\sqrt{n}\beta]^2 \\ \quad -\epsilon\lambda_{\max}(P) \geq 0 \\ \frac{1}{\tau_1} - 2\lambda_{\max}^2(P)\|K\|^2\frac{\theta}{\mu} - 2\tau_1\theta^2\|K\|^2 - \epsilon \geq 0. \end{cases} \tag{35}$$

We can easily see, that by choosing $\tau_1 = \mu/\theta^2$ and by setting $\epsilon \to 0$ the above inequalities will be equivalent to

$$\begin{cases} \frac{\mu\theta}{2} > 2\lambda_{\max}(P)\sqrt{n}\beta + 2\mu[\|A-KC\| + 2\lambda_{\max}(P)\sqrt{n}\beta]^2 \\ \theta^2 > 2\lambda_{\max}^2(P)\|K\|^2\theta + 2\mu^2\|K\|^2 \end{cases}. \tag{36}$$

Note that it is obvious that the inequalities (36) are verified for any positive constant μ by choosing sufficiently large values of θ. ∎

To summarize Lemma 1, it gives the maximum delay supported by observer (9) which enables $\hat{x}(t) \to x(t)$, once θ has been fixed according to conditions (36). To cope with a larger measurement delay, we propose in next paragraph a procedure to estimate $x(t)$, based on a cascade of high-gain observers: each observer will estimate the state at a given fraction of the output delay.

A. Cascade High Gain Observers

After proving that the convergence of the observer (9) requires a small delay, we will see that when the delay is arbitrary long and constant, a set containing a sufficient number of cascade high gain observers (7) can reconstruct the states of system (1).

Theorem 1: Let us consider system (1), then for any constant and known delay τ, there exist a sufficiently large positive constant θ and an integer m such that the last observer in (7) converges exponentially towards the system (1).

Proof: The convergence of the cascade observer will be proved step by step :

Step 1: We consider the first observer in the cascade

$$\dot{\hat{x}}_1 = A\hat{x}_1 + \phi(\hat{x}_1, u_1) - \theta\Delta^{-1}KC\left(\hat{x}_1\left(t - \frac{\tau}{m}\right) - x(t-\tau)\right). \tag{37}$$

We remark that $x(t-\tau) = x_1(t-\tau/m)$ and consequently, if we choose θ sufficiently large, and by choosing the integer m such that $m \geq (\theta^2/\mu)\tau$, then $\hat{x}_1(t)$ converges towards $x_1(t) = x(t-\tau+\tau/m) = x(t-(m-1)\tau/m)$.

Indeed, we are brought back to conditions of Lemma 1, since the delay to cope with is now τ/m, which is assumed smaller than μ/θ^2.

Step j*:* At each step ($j = 2, \ldots, m$), we estimate the delayed state $x(t-\tau+j\tau/m)$ by using the following observer:

$$\dot{\hat{x}}_j = A\hat{x}_j + \phi(\hat{x}_j, u_j) - \theta\Delta^{-1}KC\left(\hat{x}_j\left(t - \frac{\tau}{m}\right) - \hat{x}_{j-1}(t)\right). \tag{38}$$

It is not difficult to see that by considering the observation error vector $\tilde{x}_j = \hat{x}_j - x_j$, if we add and subtract the term $\theta\Delta^{-1}PKCx_{j-1}(t)$ in the previous equation, we obtain

$$\begin{aligned}\dot{\tilde{x}}_j = A\tilde{x}_j + \phi(\hat{x}_j, u_j) - \phi(x_j, u_j) - \theta\Delta^{-1}KC\tilde{x}_j\left(t - \frac{\tau}{m}\right) \\ + \theta\Delta^{-1}KC\tilde{x}_{j-1}(t).\end{aligned}$$

If we consider the following change of coordinates $\bar{x}_j = \Delta\tilde{x}_j$, we will have:

$$\begin{aligned}\dot{\bar{x}}_j = \theta(A-KC)\bar{x}_j + \Delta(\phi(\hat{x}_j, u_j) - \phi(x_j, u_j)) \\ + \theta KC\int_{t-\tau/m}^{t}\dot{\bar{x}}_j(s)ds + \theta KC\bar{x}_{j-1}. \end{aligned} \tag{39}$$

In order to prove by recurrence the convergence of the error $\bar{x}_j$, we suppose that the observation error $\bar{x}_{j-1}(t)$ converges exponentially towards zero.

Then we consider the following *Lyapunov-Krasovskii functional*

$$W_j(\bar{x}_j) = \bar{x}_j^T P\bar{x}_j + . \int_{t-\tau/m}^{t}\left(s - t + \frac{\tau}{m}\right)\|\dot{\bar{x}}_j(s)\|^2 ds \tag{40}$$

Then its time derivative satisfies the following inequality:

$$\begin{aligned}\dot{W}_j \le &-\mu\theta\|\bar{x}_j\|^2\\ &+2\bar{x}_j^T P\Delta(\phi(\hat{x}_j,u_j)-\phi(x_j,u_j))+2\theta\bar{x}_j^T PKC\bar{x}_{j-1}\\ &+2\theta\bar{x}_j^T PKC\int_{t-\tau/m}^{t}\dot{\bar{x}}_j(s)ds+\frac{\tau}{m}\|\dot{\bar{x}}_j\|^2\\ &-\int_{t-\tau/m}^{t}\|\dot{\bar{x}}_j(s)\|^2 ds. \end{aligned}\tag{41}$$

Now, by using Hölder's and Young's inequalities, we derive the following inequalities:

$$\begin{aligned}\|\dot{\bar{x}}_j\|^2 \le\ & 3\theta^2[\|A-KC\|+\sqrt{n}\beta\lambda_{\max}(P)]^2\|\bar{x}_j\|^2\\ &+3\|K\|^2\|\eta_j\|^2+3\|K\|^2\theta^2\|\bar{x}_{j-1}\|^2\end{aligned}\tag{42}$$

where, $\eta_j(t)=\theta\int_{t-\tau/m}^{t}\dot{\bar{x}}_j(s)ds$ and

$$\begin{aligned}&2\theta\bar{x}_j^T PKC\left(\bar{x}_{j-1}+\int_{t-\tau/m}^{t}\dot{\bar{x}}_j(s)ds\right)\le\frac{\mu\theta}{2}\|\bar{x}_j\|^2\\ &+\frac{4\theta\|K\|^2\lambda_{\max}^2(P)}{\mu}\|\bar{x}_{j-1}\|^2+\frac{4\|K\|^2\lambda_{\max}^2(P)}{\mu\theta}\|\eta_j(t)\|^2.\end{aligned}\tag{43}$$

Using (42) and (43), we derive

$$\begin{aligned}\dot{W}_j+\epsilon W_j \le &-\left\{\frac{\mu\theta}{2}-2\lambda_{\max}(p)\sqrt{n}\beta\right.\\ &-\frac{3\tau}{m}\theta^2[\|A-KC\|+\sqrt{n}\beta\lambda_{\max}(P)]^2\\ &\left.-\epsilon\lambda_{\max}(P)\right\}\|\bar{x}_j\|^2\\ &-\left[1-\frac{4\theta}{\mu}\lambda_{\max}^2(P)\|K\|^2\frac{\tau}{m}-3\|K\|^2\left(\frac{\tau}{m}\right)^2\theta^2\right.\\ &\left.-\epsilon\frac{\tau}{m}\right]\int_{t-\tau_1}^{t}\|\dot{\bar{x}}_j(s)\|^2ds\\ &+\left[\frac{\theta^2\|K\|^2\lambda_{\max}^2(P)}{\mu}+\frac{3\tau}{m}\|K\|^2\right]\|\bar{x}_{j-1}\|^2.\end{aligned}\tag{44}$$

Then, we can say that if $m\ge(\theta^2/\mu)\tau$ and if the following inequalities are satisfied:

$$\begin{cases}\mu\theta\ge 2\{2\lambda_{\max}(p)\sqrt{n}\beta+3\mu[\|A-KC\|+\sqrt{n}\beta\lambda_{\max}(P)]^2\\ \quad+\epsilon\lambda_{\max}(p)\}\\ \theta^2\ge 4\lambda_{\max}^2(P)\|K\|^2\theta+3\|K\|^2\mu^2+\epsilon\mu \quad \text{with} \quad \epsilon>0\end{cases}\tag{45}$$

we will have

$$\dot{W}_j\le-\epsilon W_j+\left[\frac{4\theta\|K\|^2\lambda_{\max}^2(P)}{\mu}+\frac{3\tau}{m}\|K\|^2\right]\|\bar{x}_{j-1}\|^2.\tag{46}$$

Using the comparison Lemma [14], we conclude that if $\bar{x}_{j-1}$ converges exponentially towards zero, then $\bar{x}_j$ converges also exponentially towards zero. Note that conditions (45), also ensure the convergence of the first observer $(j=1)$, then we deduce, recursively, that all observation errors converge exponentially towards zero. ■

From this, we can say that for all constant delay τ and for all θ satisfying conditions (45), there exists a number of cascade observers $m\ge(\theta^2/\mu)\tau$ such that all observation errors converge exponentially towards zero.

Remark 2: The global Lipschitz condition can be relaxed if we suppose that the vector state $x(t)$ belongs to a compact set Ω_x of $\mathbf{R}^n$. As in [15], if $\phi(x,u)$ is a C^1 function, then we replace the terms $\phi(\hat{x}_j,u)$ by $\phi(\sigma(\hat{x}_j),u)$ where σ is a saturation function defined on the set Ω_x.

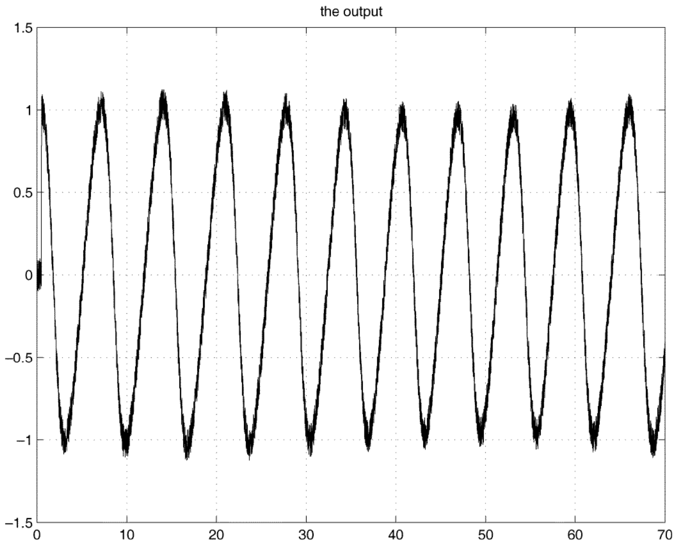

Fig. 1. Noisy delayed output measurements y.

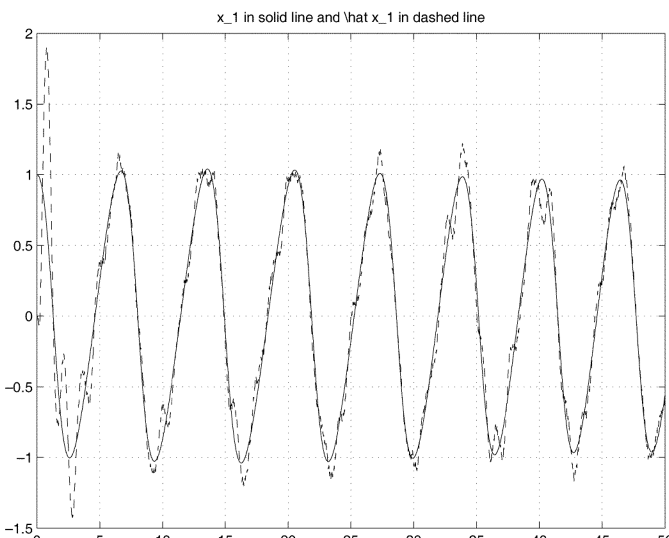

Fig. 2. Evolution of the first state for $\tau=0.25$ s with one observer.

Remark 3: The implementation of our observer can be realized by using a numerical integration of a delayed differential equation (for example the dde routine of Matlab can do this). The measures which are given by sensor are inherently delayed and they represent the delayed output. They will be used only by the first observer in the cascade.

IV. Example

To illustrate the obtained results, consider the following nonlinear system, affected by delayed measurements:

$$\begin{cases}\dot{x}_1(t)=x_2(t)\\ \dot{x}_2(t)=-x_1(t)+0.5\tanh(x_1(t)x_2(t))+x_1(t)u(t)\\ y(t)=x_1(t-\tau)\end{cases}.\tag{47}$$

The input is $u(t)=0.1\sin(0.1t)$. System (47) belongs to the considered class of triangular systems with Lipschitz nonlinearities (1).

The initial conditions for the system and for the (cascade) observer(s) have been chosen as

$$x(t)=\begin{pmatrix}0.5\\-0.5\end{pmatrix},\quad \hat{x}(t)=\begin{pmatrix}0\\0\end{pmatrix},\quad \forall t\in[-\tau,0].\tag{48}$$

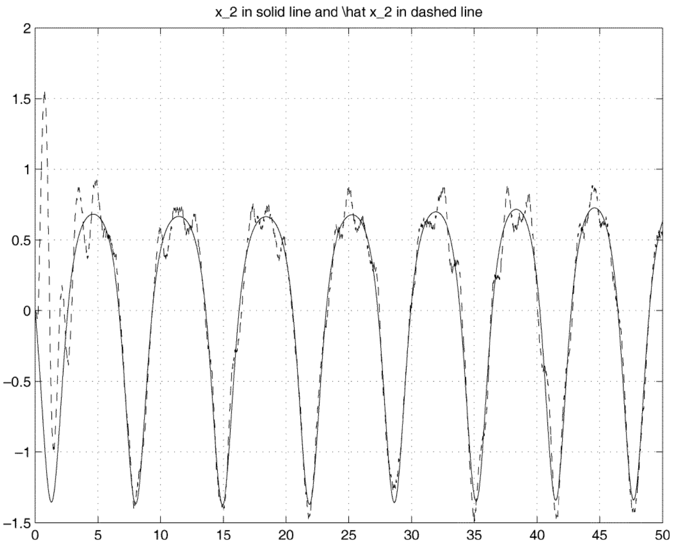

Fig. 3. Evolution of the second state for $\tau = 0.25$ s with one observer.

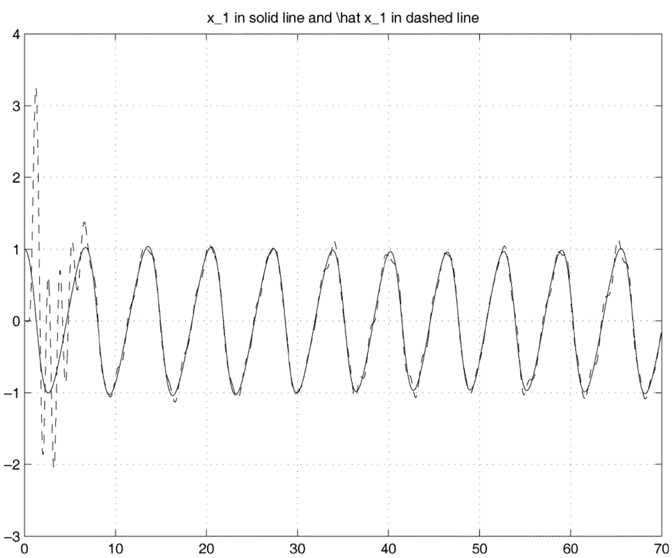

Fig. 4. Evolution of the first state for $\tau = 0.5$ s with a cascade of two observers.

Simulations have been performed using Matlab, and the integration routine *dde23*, which specifically handles delay differential systems. The high gain parameter is set to $\theta = 2$ (see Fig. 1). The following figures present the simulation results. Figs. 2 and 3 show the true states (solid) and the estimated states (dashed) when the output delay is set to 0.25 s. If the output delay is set to 0.5 s, then one observer is not sufficient to estimate the states of system (47). Figs. 4 and 5 show the true states (solid) and the estimated states (dashed) of the second observer in the cascade when two observers are used . As we can see on the presented figures, our method presents good performances even with a noise on the delayed measures.

V. Conclusion

In this technical note, a novel predictor based on high gain observer has been presented. This algorithm can be applied to a class of nonlinear uniformly observable systems. The case with unknown delay is under investigation.

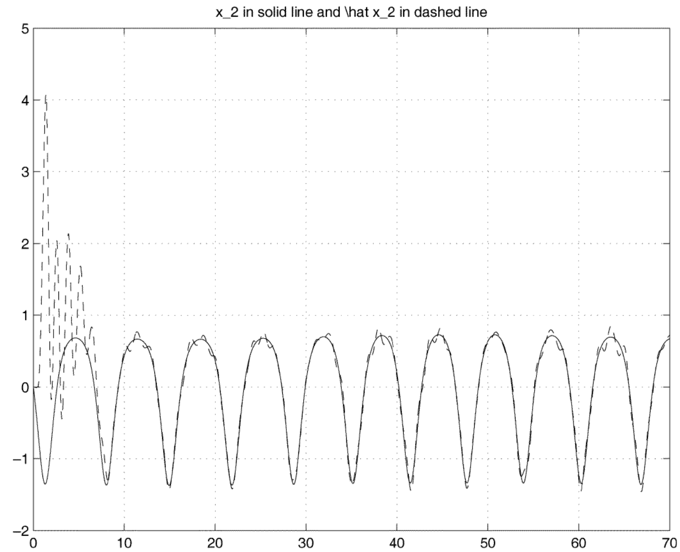

Fig. 5. Evolution of the second state for $\tau = 0.5$ s with a cascade of two observers.

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